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# Best Approximations in Cone Metric Spaces

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Abstract. Huang and Zhang defined cone metric spaces in 2007 ([1]). We shall give some results about characterization of best approximations in the cone metric spaces.

## 1. Introduction

Approximation theory has been investigated by many authors, and one can see the known book of Singer for a brief history of approximation theory ([8]). Recently, non-convex analysis has found some applications in optimization theory and approximation theory has special place in mathematics and so in non-convex analysis. In this way, Huang and Zhang defined the cone metric spaces in 2007 ([1]). At present, work on approximation theory is continued. For example, there are some works about best approximation, $\varepsilon$-best approximation and best simultaneous approximation in normed spaces, ordered normed spaces, 2-normed spaces and generalized 2-normed spaces ([2]-[7]).

Let $E$ be a real Banach space and $P$ a subset of $E$. $P$ is called a cone if
(i) $P$ is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and all non-negative real numbers $a, b$,
(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq_P$ with respect to $P$ by $x \leq_P y$ if and only if $y - x \in P$. In what follows we omit the index $P$ and write everywhere $\leq$ instead of $\leq_P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of $P$.

The cone $P$ is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$, for all $x, y \in E$.

In the following we always suppose that $E$ is a Banach space, $P$ is a cone in $E$ with $intP \neq \emptyset$ and $\leq$ is partial ordering with respect to $P$.

**Definition 1.1.** Let $X$ be a non-empty set, $E$ a Banach space and $P$ a cone in $E$. Suppose the mapping $d : X \times X \to E$ satisfies
($d_1$) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
($d_2$) $d(x, y) = d(y, x)$ for all $x, y \in X$,
($d_3$) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

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Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space ([1]).

**Example 1.2.** Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then $(X, d)$ is a cone metric space ([1]).

**Example 1.3.** Let $E = \ell^1$, $P = \{\{x_n\}_{n\in\mathbb{N}} \in E : x_n \geq 0, \text{for all n}\}$, $(X, \rho)$ a metric space and $d : X \times X \to E$ defined by $d(x, y) = \{\frac{\rho(x,y)}{2^n}\}_{n\in\mathbb{N}}$. Then $(X, d)$ is a cone metric space.

**Definition 1.4.** ([1]) Let $(X, d)$ be a cone metric space, $x \in X$ and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in $X$. Then
(i) $\{x_n\}_{n\in\mathbb{N}}$ converges to $x$ whenever for every $c \in E$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
(ii) $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
(iii) $(X, d)$ is a complete cone metric space if every Cauchy sequence is convergent.

Here we point to some elementary results of [1]. Let $(X, d)$ be a cone metric space, $P$ a normal cone with normal constant $K$, $x \in X$ and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in $X$. Then
(i) $\{x_n\}_{n\in\mathbb{N}}$ converges to $x$ if and only if $d(x_n, x) \to 0$.
(ii) Limit point of every sequence is unique.
(iii) Every convergent sequence is Cauchy.
(iv) $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
(v) If $x_n \to x$ and $x'_n \to x'$, then $d(x_n, x'_n) \to d(x, x')$ as $n \to \infty$.

**Definition 1.5.** Let $(X, d)$ be a cone metric space and $B \subseteq X$. If every sequence in $B$ has a convergent subsequence to an element of $B$, then $B$ is called a sequentially compact subset of $X$.

**Definition 1.6.** Let $(X, d)$ be a cone metric space, $G$ a non-empty subset of $X$ and $x \in X$. We say that $g_0 \in G$ is a best approximation of $x$ whenever $d(x, g_0) \leq d(x, g)$ for all $g \in G$. Then, we denote the set of all best approximations of $x$ in $G$ by $P_G(x)$.

**Definition 1.7.** Let $(X, d)$ be a cone metric space and $G$ a non-empty subset of $X$. We say that $G$ is a Chebyshev subset of $X$ if $P_G(x)$ is a singleton subset of $G$ for all $x \in X$. Also, we say that $G$ is a quasi Chebyshev subset of $X$ if $P_G(x)$ is sequentially compact subset of $X$ for all $x \in X$.

**Definition 1.8.** Let $X$ be a real vector space, $(X, d)$ a cone metric space and $G$ a non-empty subset of $X$. We say that $G$ is a pseudo Chebyshev subset of $X$ if there is no $x \in X$ such that $P_G(x)$ contains infinitely many linearly independent elements.

**Example 1.9.** Let $E = \ell^1$, $P = \{\{x_n\}_{n\in\mathbb{N}} \in E : x_n \geq 0, \text{for all } n\}$, $(X, \|.\|)$ a normed space, $G$ a quasi Chebyshev subset of $X$ and $d : X \times X \to E$ defined by $d(x, y) = \{\frac{\|x-y\|}{2^n}\}_{n\in\mathbb{N}}$. Then, $G$ is a quasi Chebyshev subset of $(X, d)$.

## 2. Main Results

Now we are ready to state our main results.

**Lemma 2.1.** *Let $(X, d)$ be a cone metric space, $G$ a non-empty subset of $X$, $g_0 \in G$ and $x \in X$. Then, $g_0 \in P_G(x)$ if and only if there exists a function $f : X \to E$ such that $f(g_0) = d(x, g_0)$, $f_{g_0}(g) := f(g) - f(g_0) \in P$ and $f_d(g) := d(x, g) - f(g) \in P$ for all $g \in G$.*

*Proof.* First suppose that there exists a function $f : X \to E$ such that $f(g_0) = d(x, g_0)$, $f_{g_0}(g) \in P$ and $f_d(g) \in P$ for all $g \in G$. Since $f_{g_0}(G) \subseteq P$ and $f_d(G) \subseteq P$, $f(g_0) \leq f(g)$ and $f(g) \leq d(x, g)$ for all $g \in G$. Thus, $d(x, g_0) = f(g_0) \leq f(g) \leq d(x, g)$ for all $g \in G$. Hence, $g_0 \in P_G(x)$.
For the converse part, define $f : X \to E$ by $f(t) = d(x, t)$. Then, $f(g_0) = d(x, g_0)$, $f_{g_0}(G) \subseteq P$ and $f_d(G) = \{0\} \subseteq P$. □

**Theorem 2.2.** *Let $(X, d)$ be a cone metric space, $G$ a non-empty subset of $X$ and $x \in X$. Then, $M \subseteq P_G(x)$ if and only if there exists a function $f : X \to E$ such that $f(m) = d(x, m)$, $f_m(G) \subseteq P$ and $f_d(G) \subseteq P$ for all $m \in M$.*

*Proof.* First suppose that there is a function $f : X \to E$ such that $f(m) = d(x, m)$, $f_m(G) \subseteq P$ and $f_d(G) \subseteq P$ for all $m \in M$. Then by Lemma 2.1, $m \in P_G(x)$ for all $m \in M$. Hence, $M \subseteq P_G(x)$.
For the converse part, take an arbitrary element $m_1 \in M$. By Lemma 2.1, there is a function $f : X \to E$ such that $f(m_1) = d(x, m_1)$, $f_{m_1}(G) \subseteq P$ and $f_d(G) \subseteq P$. Let $m \in M$. Then, $f_{m_1}(m) \in P$ and $f_d(m) \in P$. Since $m \in P_G(x)$, $d(x, m) \leq d(x, m_1) \leq f(m) \leq d(x, m)$. Hence, $f(m) = d(x, m)$. Also,

$$f_m(g) = f(g) - f(m) = f(g) - d(x, m) = f(g) - d(x, m_1) = f_{m_1}(g) \in P$$

for all $g \in G$. Therefore, $f$ is the desired function. □

**Corollary 2.3.** *Let $(X, d)$ be a cone metric space and $G \subseteq X$. Then, $G$ is a Chebyshev subset of $X$ if and only if there don't exist $x \in X$, distinct elements $g_1, g_2 \in G$ and a function $f : X \to E$ such that $f(g_i) = d(x, g_i)$, $f_{g_i}(G) \subseteq P$ and $f_d(G) \subseteq P$ for $i = 1, 2$.*

**Theorem 2.4.** *Let $(X, d)$ be a cone metric space and $G \subseteq X$. Then, $G$ is quasi Chebyshev subset of $X$ if and only if there don't exist $x \in X$, a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $G$ without a convergent subsequence and a function $f : X \to E$ such that $f(g_n) = d(x, g_n)$, $f_{g_n}(G) \subseteq P$ and $f_d(G) \subseteq P$ for all $n \in \mathbb{N}$.*

*Proof.* First suppose that there exist $x \in X$, a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $G$ without a convergent subsequence and a function $f : X \to E$ such that $f(g_n) = d(x, g_n)$, $f_{g_n}(G) \subseteq P$ and $f_d(G) \subseteq P$ for all $n \in \mathbb{N}$. Then by Theorem 2.2, $g_n \in P_G(x)$ for

all $n \in \mathbb{N}$. It follows that $P_G(x)$ is not sequentially compact.
For the converse part, suppose that $G$ is not quasi Chebyshev subset of $X$. Then, there exist $x \in X$ and a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $P_G(x)$ without a convergent subsequence. By Theorem 2.2, there exists $f : X \to E$ such that $f(g_n) = d(x, g_n)$, $f_{g_n}(G) \subseteq P$ and $f_d(G) \subseteq P$ for all $n \in \mathbb{N}$. □

The proof of the following Theorem is similar to that of Theorem 2.4.

**Theorem 2.5.** *Let $X$ be a real vector space, $(X, d)$ a cone metric space and $G \subseteq X$. Then, $G$ is pseudo Chebyshev subset of $X$ if and only if there don't exist $x \in X$, infinitely many linearly independent elements $\{g_n\}_{n\in\mathbb{N}}$ in $G$ and a function $f : X \to E$ such that $f(g_n) = d(x, g_n)$, $f_{g_n}(G) \subseteq P$ and $f_d(G) \subseteq P$ for all $n \in \mathbb{N}$.*

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